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On König's Theorem for Infinite Bipartite Graphs

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1. INTRODUCTION

Let X and Y be disjoint sets. Let $|S|$ denote the cardinal of the set S. Let $Z = X \cup Y$ be the set of all points, $X \times Y$ the set of all *edges* and $\Delta \subset X \times Y$. A subset δ of Δ is said to be *dependent* if it contains at least two edges with a common endpoint. Otherwise δ is said to be *independent*.

Let there be a graph $G = (Z, \Delta)$. A graph thus defined is said to be *bipartite*. $Z' \subset Z$ is said to be a *separating set* in G if every edge in Δ has an endpoint in Z'.

A subset A of B is *c-maximal* in B with respect to property P if there does not exist a subset A' of B having property P and such that $|A| < |A'|$.

König's Theorem for finite-bipartite graphs states:

If there does not exist a separating set in G of cardinality less than α, then there exist α independent edges in G.

This theorem has since been extended to infinite bipartite graphs with c-maximal sets of independent edges (Theorem 7.9.3 in [7]).

The purpose of this paper is to supply an independent proof for infinite bipartite graphs. Also, by using a rather simple argument, the hypothetical nature of the extension in [7] is disposed of. The only assumptions which are made are König's Theorem for finite graphs and the axiom of choice.

2. THE PROOF

Case 1.

$$\alpha \geqslant \aleph_0 .$$

The class of all independent subsets of Δ is partially ordered by inclusion. Therefore, by Zorn's lemma, Δ has an inclusion-maximal independent subset δ_0.

Let $z \in \kappa$ if there exists z' such that either $(z, z') \in \delta_0$ or $(z', z) \in \delta_0$. If κ is not a separating set in G, then there exist $x, y \notin \kappa$ such that $(x, y) \in \Delta$. But the construction of κ implies $(x, y) \notin \delta_0$ and hence $\delta_0 \cup \{(x, y)\}$ is independent, contrary to the maximal property of δ_0. Then κ is a separating set in G and $|\kappa| = 2 |\delta_0|$. Then

$$\alpha \leqslant |\kappa| = 2 |\delta_0|. \tag{1}$$

α was assumed infinite and hence $\alpha \leqslant |\delta_0|$. This proves case 1.

Case 2.

$$\alpha < \aleph_0, \qquad |\Delta| \geqslant \aleph_0.$$

Equation (1) is still valid in Case 2; so that κ and δ_0 are either both finite or both infinite. If they are both infinite, then the theorem is already proved because α is finite by assumption. We may therefore assume the existence of a finite separating set $S = \{z_1, ..., z_\alpha\}$ in G.

Put:

$$\Delta_x = (\{x\} \times Y) \cap \Delta, \qquad x \in X;$$
$$\Delta_y = (X \times \{y\}) \cap \Delta, \qquad y \in Y.$$

Let $z \in Z$. Whenever $|\Delta_z| < \aleph_0$, define $\Delta_z' = \Delta_z$; whenever $|\Delta_z| \geqslant \aleph_0$, define Δ_z', so that

$$\Delta_z' \subset \Delta_z \qquad \text{and} \qquad |\Delta_z'| = \alpha, \tag{2}$$

otherwise arbitrary.

Put

$$\Delta' = \bigcup_{i=1}^{n} \Delta'_{z_i}.$$

Then $|\Delta'| < \aleph_0$. Put $G' = (Z, \Delta')$. Then $G' \subset G$ and G' is finite and bipartite.

Let now Σ be a separating set in G'. Suppose $|\Sigma| < \alpha$. If $|\Delta_{z_i}| \geqslant \aleph_0$ for some $z_i \in Z$, then $z_i \in \Sigma$ because of (2). It follows that Σ is a separating set in G, which is a contradiction because $|\Sigma| < \alpha$. Then there is no separating set in G' of cardinality less than α, and hence by the theorem for finite-bipartite graphs, there exist α independent edges in G' and hence in G, which proves Case 2.

3. König's Theorem as an Equality

Let $G = (Z, \Delta)$. Put $A(G) = \{\delta : \delta \subset \Delta, \delta \text{ independent}\}$. Let $Q(G)$ be the class of all separating sets in G.

G is said to be of *finite rank* if there is no infinite set of independent edges in G.

We note that if $\delta_1, \delta_2 \in A(G)$ and $2|\delta_1| < |\delta_2|$, then there exists $(x, y) \in \delta_2$ such that $\delta_1 \cup \{(x, y)\} \in A(G)$. Now let G be of *infinite rank* (i.e., not of finite rank).

Define $A_0(G) = \{\delta : \delta \in A(G)$ and there exists no $\delta_1 \neq \delta, \delta_1 \in A(G)$ such that $\delta \subset \delta_1\}$. By Zorn's lemma, $A_0(G) \neq \varnothing$. Let $\delta_1, \delta_2 \in A_0(G)$. We show that $|\delta_1| = |\delta_2|$. Suppose $|\delta_1| < |\delta_2|$.

Case 1. $|\delta_1| < \aleph_0$. Then there exists $\delta_3 \in A(G)$ such that

$$2|\delta_1| < |\delta_3|.$$

Case 2. $|\delta_1| \geqslant \aleph_0$. Then

$$2|\delta_1| < |\delta_2|.$$

In both cases, $\delta_1 \notin A_0(G)$, a contradiction. Then $|\delta_1| = |\delta_2|$. It follows that all elements of $A_0(G)$ have equal cardinality as subsets of Δ. Define $m(G) = |\delta|$, where $\delta \in A_0(G)$; $D = \{|\delta| : \delta \in A(G)\}$. Clearly, $m(G)$ is the largest cardinal in D. Put $K = \{|q| : q \in Q\}$. Define $M(G)$ to be the least cardinal in K. Since $A_0(G) \neq \varnothing$, it follows immediately from the result in Section 2 that

$$m(G) = M(G). \tag{3}$$

Thus (3) holds for all bipartite graphs.

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